

Distinguished properties of the gamma process, and related topics

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Abstract

We study fundamental properties of the gamma process and their relation to various topics such as Poisson–Dirichlet measures and stable processes. We prove the quasi-invariance of the gamma process with respect to a large group of linear transformations. We also show that it is a renormalized limit of the stable processes and has an equivalent sigma-finite measure (quasi-Lebesgue) with important invariance properties. New properties of the gamma process can be applied to the Poisson–Dirichlet measures. We also emphasize the deep similarity between the gamma process and the Brownian motion. The connection of the above topics makes more transparent some old and new facts about stable and gamma processes, and the Poisson–Dirichlet measures.

Keywords. Lévy processes, gamma process, stable processes, Poisson–Dirichlet measures, multiplicative quasi-invariance, quasi-Lebesgue measure, Markov–Krein identity.

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1 Introduction

The purpose of this work is to link various questions from the probability theory, combinatorics, and representation theory, which relate to the gamma process and the Poisson–Dirichlet measures, and which were not regarded as a

single whole. Since in spite of many aspects of our work its main object is the gamma process, first we would like to present some fundamental properties of the classical gamma process and its role for further considerations.

The standard gamma distribution (e.g. see [14]) with parameter $\alpha > 0$ is a distribution on the positive half-line with the following density:

$$\frac{u^{\alpha-1}e^{-u}}{\Gamma(\alpha)}, \quad u > 0.$$

Random variable with the gamma distribution is called a gamma variable. The classical gamma process $(\gamma_t, t \geq 0)$ is an increasing stochastic process with independent homogeneous increments such that γ_t has the gamma distribution with parameter t for each $t > 0$.

The following fundamental properties of this process have led us to more general developments and definitions discussed below, and we suggest that the reader comes back to them as a “toy model” when reading more general later parts of our paper.

1. Decomposition property: If two variables ξ, ϕ are independent and have standard gamma distributions, then $\xi + \phi$ and $\frac{\xi}{\xi + \phi}$ are also independent. Moreover, this is a *characteristic property* of the gamma distribution (Lukacs’ theorem, see Section 4 and [14]). Namely, if x_1, x_2 are independent positive random variables, and $x_1 + x_2$ and $\frac{x_1}{x_1 + x_2}$ are also independent, then x_1 and x_2 are common multiples of standard gamma variables with some parameters.

The first property easily implies

2. Independence property: For each $t > 0$, the process $(\gamma_u/\gamma_t, u \leq t)$ is independent on $(\gamma_v, v \geq t)$.

From property 2 one can deduce the following

3. Quasi-invariance property: For each $t > 0$ and $a > 0$ the law of the process $(\frac{\gamma_u}{1+a}, u \leq t)$ is equivalent to that of $(\gamma_u, u \leq t)$ with Radon–Nikodym density $(1+a)^t \cdot \exp(-a\gamma_t)$.

This property will lead us to a very important quasi-invariance of the law of the gamma process which was discovered in [32] in a different way and was used for the representation theory (see Section 5).

- 4. Gamma distribution as a limit of stable distributions:** It was noticed that for a small index α the stable distribution could be renormalized in such a way that the limit when α tends to zero is the gamma distribution [6, 26]. This fact leads us to an important conclusion (obtained also in another way in [34]) that the law of the gamma process is a limit of renormalized stable processes when $\alpha \rightarrow 0$ (see Section 9).

The last property allows us to consider the gamma process as an antithesis to the Brownian motion: both correspond to the extreme values of the parameter $\alpha \in [0, 2]$, and in between we have stable processes which can be considered as a deformation from the Brownian motion to the gamma process. Moreover, we can see many strikingly similar properties, for example:

- 1'. Classical Bernstein's characteristic property of Gaussian measures (components of both pairs of variables (X, Y) and $(X + Y, X - Y)$ are independent if and only if the distributions of X and Y are Gaussian) is similar to our property 1.
- 2'. Let $(B_t, t \geq 0)$ be a Brownian motion. For every $t > 0$ the Brownian bridge $(B_u - \frac{u}{t}B_t, u \leq t)$ is independent on $(B_v, v \geq t)$, which is similar to our property 2.
- 3'. For every $b \in \mathbb{R}$ and every $t > 0$ the law (a measure in the space of realizations) of the process $(B_u + bu, u \leq t)$ is equivalent to that of $(B_u, u \leq t)$ with Radon–Nikodym density $\exp(bB_t - b^2t/2)$. The large group of symmetries of the gamma process is a multiplicative group of hyperbolic rotations (see Section 5). The law of the Brownian motion has a large orthogonal group of symmetries. Stable processes perhaps also have such large groups of symmetries, but they consist of non-linear transformations, and can be considered as a homotopy between hyperbolic and orthogonal rotations.

In view of this parallelism, it would be quite interesting to connect better the Brownian motion and the gamma process, and one such attempt is recorded in the Appendix, where we have gathered further properties of the gamma process which are not so close to the main context of the paper.

- 5. Generalized stability property:** In view of the previous property stating that the gamma process is a limit of *stable* processes it is natural to ask what kind of stability do the gamma distribution and the gamma

process have? In Section 10 we introduce a notion of generalized stability for *sigma-finite measures*, and show that the Lebesgue measure on the real line and the quasi-Lebesgue measure which is a sigma-finite measure equivalent to the law of the gamma process are also stable in this new sense.

Now we present our general framework and the list of topics touched on in the paper. The first one is the theory of Poisson–Dirichlet measures from the point of view of the gamma process. More general, we consider the theory of *Lévy processes* (a class of processes with independent values) without Gaussian component. The laws of such processes admit a canonical decomposition into a so-called *conic part*, i.e. a measure on the cone of positive series, and a standard product measure on sequences of points of the base space. This decomposition is of the most general character. It appeared first in [11], but the original proof was rather complicated and shaded the key relations with measures on positive series and a special role of stable and gamma processes, while our proof is based on very general and simple considerations.

This decomposition leads us to the theory of measures on positive series, i.e. the theory of *random positive series*, and their projections on the simplex of series with unit sum. Particularly, we deal with the *Poisson–Dirichlet measures* $PD(\theta)$ and their generalizations which were studied by Kingman [18], Vershik–Shmidt [33], Pitman–Yor [25] and their followers, and have numerous applications in combinatorics (e.g. [33, 13]), number theory [3, 30], mathematical biology (e.g. [19], see a detailed survey in [8]), etc. See also some interesting discussion on the Poisson–Dirichlet measures in most recently published book [4]. These random series may be roughly characterized as “the most random convergent series” — a kind of white noise on convergent series.

The key feature in this work is the quasi-invariance of the gamma measure (the law of the gamma process) with respect to an infinite-dimensional group of multipliers. It was first discovered and used in the works of Gelfand–Graev–Vershik [31, 32, 12] on the representation theory of current groups, more exactly, of the group $SL(2, F)$, where F is an algebra of functions on a manifold. This property of the gamma measure followed from rather indirect considerations. The same considerations prompted the existence of an equivalent σ -finite measure which is invariant under multiplications by non-negative functions with zero integral of logarithm. We call this measure *quasi-Lebesgue*, because of its key property which is an infinite-dimensional

generalization of the well-known property of the finite-dimensional Lebesgue measure. We mean invariance of the Lebesgue measure under the action of diagonal matrices with determinant 1 (a Cartan subgroup). In this work we prove this quasi-invariance directly, starting from the characteristic functional (the Laplace transform) of the gamma measure, and construct explicitly the corresponding σ -finite measure. The same quasi-invariance implies new symmetry properties of the Poisson–Dirichlet measures with respect to Markovian transformations.

However, the most important is the following link outlined in [34]. Both the gamma measure and the quasi-Lebesgue measure are weak limits of α -stable measures (the laws of α -stable processes) when α goes to zero. In terms of the Poisson–Dirichlet measures this fact was proved in [25]. More exactly, Pitman–Yor [25] define a two-parameter family of measures $\text{PD}(\alpha, \theta)$ on the simplex of positive series with unit sum and show that they converge to the Poisson–Dirichlet measures $\text{PD}(\theta)$ when $\alpha \rightarrow 0$. It turns out that this convergence follows from the convergence of renormalized α -stable measures to the gamma measure when $\alpha \rightarrow 0$. This convergence is a “commutative” analogue of the key fact discovered in [31] which deals with the limit (more exactly, the derivative with respect to the parameter in 0 which corresponds to a renormalization before taking the limit) of positive definite spherical functions of the complementary series of $\text{SL}(2, R)$ when the parameter tends to a critical value (the so-called canonical state). Thus the Poisson–Dirichlet measures are directly related to the representation theory.

Another corollary of the quasi-invariance of the gamma measure allows to obtain easily the *Markov–Krein identity* which in our context relates the distribution of a linear functional on the gamma process and the distribution of the same functional on the normalized gamma process.

The paper is organized as follows.

In Section 2 we define a general class of Lévy processes. The main properties of these processes are studied in Section 3. Though in the sequel we deal only with the stable and gamma processes, the Decomposition Theorem 1 is proved by so general and natural considerations that we give it in the most general form that does not complicate the argument. This theorem states that the law of a Lévy process is the product of the conic part (the measure on the cone of positive convergent series) and a product measure on sequences of points of the base space. In fact, our proof applies to even more general processes. Theorem 2 is a characterization of measures on the

cone of positive convergent series which are obtained as conic parts of Lévy processes (so-called measures of *product type*). In some sense, these measures enjoy the greatest possible independence of coordinates. Note that passing from the simplex to the cone (a kind of poissonization) simplifies many questions. For example, characterization of the conic parts of Lévy processes is simpler than characterization of the simplicial parts.

Section 4 contains definition and basic properties of the gamma process and the Poisson–Dirichlet measures which are the simplicial parts of the gamma process.

In Section 5 we prove the key property of the gamma measure, namely its quasi-invariance with respect to a group of multipliers (Theorem 3). This group is rather wide and consists of all non-negative measurable functions with summable logarithm. Thus the gamma measure is a multiplicative analogue of the Wiener measure which is quasi-invariant with respect to a wide group of additive shifts. This parallel should certainly be considered more thoroughly. It is not known whether the measure with such supply of preserving transformations is unique. We give a partial counter-example for a smaller group of transformations. As to the quasi-invariance under the whole group, the gamma measure seems to be the unique measure enjoying this property.

In Section 6 we apply quasi-invariance of the gamma process to obtain the corresponding property of the Poisson–Dirichlet measures.

In Section 7 we introduce a σ -finite measure which is already invariant (projective invariant) under the same group of multipliers. It is called *quasi-Lebesgue* since it generalizes a well-known property of the Lebesgue measure in \mathbb{R}^n .

Section 8 is devoted to definitions and basic properties of the stable processes. The simplicial parts of these properties are related to the two-parameter Poisson–Dirichlet measures $PD(\alpha, \theta)$.

In Section 9 we prove the statement suggested in [34] that the gamma process is a weak limit of renormalized stable processes. As was noted above, this fact is related to the representation theory [31, 32, 12].

In Section 10 we give a new definition of stability which can be applied to σ -finite measures as well (unlike the classical definition). According to this definition, the quasi-Lebesgue measure is zero-stable.

Finally, in Sections 11 and 12 we deal with the *Markov–Krein transform* and its generalizations. We present a new probabilistic interpretation of this transform: formulae of this kind relate the distribution of a linear functional

on the process with the distribution of the same functional on the normalized process. This interpretation sets the same question for general Lévy processes.

In the Appendix we try to trace some further connection of our topics, namely, some new and unexpected links with Brownian motion.

The topics touched upon in this paper stimulate many new problems, only a small part of which is mentioned above.

2 Definition of Lévy processes on arbitrary spaces

Let (X, ν) be a standard Borel space with a non-atomic finite non-negative measure ν , and let $\nu(X) = \theta$ be the total charge of ν . We denote by

$$D = \left\{ \sum z_i \delta_{x_i}, \ x_i \in X, \ z_i \in \mathbb{R}, \ \sum |z_i| < \infty \right\}$$

a real linear space of all finite real discrete measures on X .

Consider a class of measures Λ on the half-line \mathbb{R}_+ satisfying the following conditions,

$$\Lambda(0, \infty) = \infty, \tag{1}$$

$$\Lambda(1, \infty) < \infty, \tag{2}$$

$$\int_0^1 s d\Lambda(s) < \infty, \tag{3}$$

$$\Lambda(\{0\}) = 0. \tag{4}$$

Let F_Λ be the infinitely divisible distribution with Lévy measure Λ , i.e. the Laplace transform ψ_Λ of F_Λ is given by

$$\psi_\Lambda(t) = \exp \left(- \int_0^\infty (1 - e^{-ts}) d\Lambda(s) \right).$$

Definition 1 *A homogeneous Lévy process on the space (X, ν) with Lévy measure Λ satisfying (1)–(4) is a generalized process on D whose law $P_\Lambda = P_\Lambda(\nu)$ has the Laplace transform*

$$\mathbb{E} \left[\exp \left(- \int_X a(x) d\eta(x) \right) \right] = \exp \left(\int_X \log \psi_\Lambda(a(x)) d\nu(x) \right), \tag{5}$$

where a is an arbitrary non-negative bounded Borel function on X .

The correctness of this definition is guaranteed by the following explicit construction (see [20, chapter 8]). Consider a Poisson point process on the space $X \times \mathbb{R}_+$ with mean measure $\nu \times \Lambda$. We associate with a realization $\Pi = \{(X_i, Z_i)\}$ of this process an element

$$\eta = \sum_{(X_i, Z_i) \in \Pi} Z_i \delta_{X_i} \in D. \quad (6)$$

Then η is a random discrete measure obeying the law P_Λ . Note that if Λ is a δ -measure δ_z for some $z \in \mathbb{R}_+$, then Π is the Poisson process on the set $X \times \{z\}$ (which we identify with X) with mean measure ν , and the corresponding random element η is a measure that has equal charges z at the points of this process. Thus a Lévy process with an arbitrary measure Λ is a continual convolution of independent Poisson processes on X corresponding to different levels (charges).

It follows that the law P_Λ of the Lévy process is concentrated on the cone $D^+ = \{\sum z_i \delta_{x_i} \in D : z_i > 0\} \subset D$ consisting of all finite positive discrete measures on X . The conditions (1)–(4) imposed on the measure Λ have the following meaning: (1) implies that the random measure η has an infinite number of atoms; (2) together with (3) guarantees that η is a finite measure, i.e. the sum of charges is finite; finally, (4) means that our Lévy process has no Gaussian component.

Remarks. 1. Our definition of the Lévy process is closely related to the notion of completely random measure, see [17], [20, chapter 8].

2. If $X = \mathbb{R}_+$ and ν is the Lebesgue measure on \mathbb{R}_+ , we recover an ordinary definition of a *subordinator* (a process with stationary positive independent increments) corresponding to the Lévy measure Λ .

3. It is easy to see that $P_\Lambda(\nu) = P_{\theta\Lambda}(\nu/\theta)$, i.e. we may consider only normalized parameter measures ν . Thus *in the sequel we assume* $\nu(X) = 1$.

3 Decomposition theorem for Lévy processes and measures of product type on the cone

Consider the cone

$$C = \{z = (z_1, z_2, \dots) : z_1 \geq z_2 \geq \dots \geq 0, \sum z_i < \infty\} \subset l^1.$$

We now define a special class of measures on C indexed by infinitely divisible distributions on the half-line. Fix an integer $n \in \mathbb{N}$ and a probability vector

$p = (p_1, \dots, p_n)$ (i.e. a vector p with $p_1, \dots, p_n > 0$ and $p_1 + \dots + p_n = 1$). Consider a sequence ξ_i of i.i.d. variables such that $P(\xi_i = k) = p_k$ for $k = 1, \dots, n$. For $Q = (Q_1, Q_2, \dots) \in C$, denote by $\Sigma_k^{(p)} = \Sigma_k^{(p)}(Q)$ the random sum $\Sigma_k^{(p)} = \sum_{i:\xi_i=k} Q_i$. Let Q be a random series with distribution \varkappa on C such that the distribution F of the sum $\sum Q_i$ is infinitely divisible.

Definition 2 *We say that a series Q (and its distribution \varkappa) is of product type, if for each $n \in \mathbb{N}$ and each probability vector p the sums $\Sigma_1^{(p)}, \dots, \Sigma_n^{(p)}$ are independent and $\Sigma_k^{(p)}$ obeys the law F^{*p_k} .*

We define a map $T : D^+ \rightarrow C \times X^\infty$ by

$$T\eta = ((Q_1, Q_2, \dots), (X_1, X_2, \dots)), \quad \text{if} \quad \eta = \sum Q_i \delta_{X_i}. \quad (7)$$

Definition 3 *Let P be a distribution on the space D^+ , and let η be a random process obeying the law P . The random sequence of charges Q_1, Q_2, \dots is called the conic part of the process η , and its distribution on the cone C is called the conic part of the law P .*

Note that in view of representation (6) the conic part of the Lévy process with Lévy measure Λ is just the ordered sequence of points of the Poisson process on \mathbb{R}_+ with mean measure Λ . Thus the conic part depends only on Λ and not on the (X, ν) . In fact, the following theorem shows that studying the Lévy process may be essentially reduced to studying its conic part, since the construction of the process includes the parameter measure in a trivial way. This fundamental property of homogeneous Lévy processes is a particular case of the representation theorem first proved in [11]. We present here a simpler proof of this fact. The key point of our proof is the following lemma. Let (X, ν) be a standard Borel space with continuous probability measure ν . Denote $X^k = X \times \dots \times X$ (k factors), $\nu^k = \nu \times \dots \times \nu$ (k factors) and let ν_{diag} be the image of ν under the diagonal map $x \rightarrow (x, \dots, x)$.

Lemma 1 *Let τ be some continuous probability measure on X^k . If for each measure preserving transformation L of (X, ν) , the transformation $L^k = L \times \dots \times L$ (k factors) preserves τ , then τ is a convex combination of ν^k and ν_{diag} .*

Proof. For simplicity, assume $k = 2$, the general case being quite similar. The diagonal $\Delta = \{(x, x), x \in X\}$ is obviously an invariant subset for the group $G = \{L \times L\}$, where L runs over the set of all ν -preserving transformations of the space X . Thus it suffices to show that if τ is concentrated on the set $(X \times X) \setminus \Delta$, then $\tau = \nu \times \nu$, and if τ is concentrated on Δ , then $\tau = \nu_{diag}$.

In the first case let $\xi_n = \{A_i\}_{i=1}^{2^n}$ be an arbitrary partition of the space X into 2^n sets of equal ν -measure $1/2^n$. Denote by ξ_n^2 the corresponding partition of the space $X \times X$, i.e. $\xi_n^2 = \{A_{ij}\}$, where $A_{ij} = A_i \times A_j$. The group G acts transitively on the set of non-diagonal elements of ξ_n^2 . Thus all non-diagonal elements have equal τ -measure. Denote $Y_n = (X \times X) \setminus \bigcup (A_i \times A_i)$ and $\varepsilon_n = \tau(Y_n)$. Since τ is concentrated on $(X \times X) \setminus \Delta$, we have $\varepsilon_n \rightarrow 1$ as $n \rightarrow \infty$. Considering finer partitions and using the above argument, we obtain that for each $k \in \mathbb{N}$, if a rectangle $A \times B \subset Y_n$ and $\nu(A) = \nu(B) = \nu(Y_n)/2^k$, then $\tau(A \times B) = \varepsilon_n/4^k$. But then the restriction of τ on the set Y_n equals $\varepsilon_n \cdot (\nu \times \nu)$. Letting $n \rightarrow \infty$, we obtain $\tau = \nu \times \nu$.

In the second case, identifying the diagonal Δ with X , we obtain that τ is a measure on X which is invariant under all ν -preserving transformations, hence obviously $\tau = \nu$. \square

Theorem 1 *Let $\eta = \sum Q_i \delta_{X_i}$ be a homogeneous Lévy process on the space (X, ν) with Lévy measure Λ . Then $TP_\Lambda = \kappa_\Lambda \times \nu^\infty$, i.e. X_1, X_2, \dots is a sequence of i.i.d. random variables with common distribution ν , and this sequence is independent of the conic part $\{Q_i\}_{i \in \mathbb{N}}$.*

Proof. Let $L : X \rightarrow X$ be a ν -preserving transformation of X . This transformation acts on the space D by substituting coordinates, i.e. $\sum z_i \delta_{x_i} \mapsto \sum z_i \delta_{Lx_i}$, and it is clear that the law P_Λ of a homogeneous Lévy process on (X, ν) is invariant under L . Denote by P_Λ^z the conditional measure of P_Λ given the conic part equal to $z \in C$. The transformation L acts “fibre-wise”, i.e. it does not change the conic part, hence L preserves almost all conditional measures P_Λ^z . In particular, if we denote by $(P_\Lambda^z)_k$ the conditional distribution of the first k points X_1, \dots, X_k on the space X^k , then the transformation L^k preserves $(P_\Lambda^z)_k$.

Now it follows from Lemma 1 that for almost all z $(P_\Lambda^z)_k = \nu^k$ for all k , i.e. $P_\Lambda^z = \nu^\infty$, and Theorem 1 follows. \square

Theorem 2 *The measure \varkappa on the cone C is the conic part of some Lévy process P_Λ with Lévy measure Λ satisfying (1)–(4) if and only if it is of product type with $F = F_\Lambda$.*

Proof. Given fixed $n \in \mathbb{N}$ and a probability vector $p = (p_1, \dots, p_n)$, consider a partition $X = A_1 \cup \dots \cup A_n$ of the space X such that $\nu(A_k) = p_k$, $k = 1, \dots, n$. Let X_1, X_2, \dots be the sequence of i.i.d. variables with common distribution ν and assume $\xi_i = k$, if $X_i \in A_k$. Then the random variables ξ_i form a sequence of i.i.d. variables, and $P(\xi_i = k) = p_k$. Consider a random process

$$\eta = \sum Q_i \delta_{X_i}, \quad (8)$$

where the sequence Q_1, Q_2, \dots is independent of $\{X_i\}$ and obeys the law \varkappa . Let $\Sigma_k^{(p)} = \sum_{i:\xi_i=k} Q_i$. It is easy to see that for arbitrary $t_1, \dots, t_n > 0$

$$\mathbb{E} \left[\exp \left(- \sum_{k=1}^n t_k \Sigma_k^{(p)} \right) \right] = \mathbb{E} \left[\exp \left(- \int_X a(x) d\eta(x) \right) \right], \quad (9)$$

where a is a step function such that $a(x) = t_i$, if $x \in A_i$.

Now let \varkappa be the conic part of the law P_Λ of some Lévy process. Then, by Theorem 1, the process η defined by (8) obeys P_Λ , and it follows from the Laplace transform formula (5) that the right-hand side of (9) equals

$$\prod_{i=1}^n \psi_\Lambda(t_i)^{\nu(A_i)} = \prod_{i=1}^n \psi_\Lambda(t_i)^{p_i}.$$

Since the left-hand side of (9) is the Laplace transform of the common distribution of the variables $\Sigma_1^{(p)}, \dots, \Sigma_n^{(p)}$, we obtain that $\Sigma_1^{(p)}, \dots, \Sigma_n^{(p)}$ are independent, and $\Sigma_k^{(p)}$ obeys the law F^{*p_k} , i.e. \varkappa is of product type.

Conversely, let \varkappa be a measure of product type corresponding to an infinitely divisible law F . Define a random process η on an arbitrary measurable space (X, ν) satisfying the conditions of Definition 1 by (8). The above argument shows that η satisfies (5) with Λ equal to the Lévy measure of F for all positive step functions a , and one can easily extend this to all bounded positive Borel functions by continuity. Thus η obeys P_Λ , and \varkappa is the conic part of P_Λ . \square

The strong law of large numbers for Poisson processes (see [20, 4.55]) implies the following result on the asymptotic behavior of the vectors $Z = (Z_1, Z_2, \dots) \in C$ obeying the law \varkappa_Λ .

Proposition 1 ([18]) *Let $m_\Lambda(t) = \Lambda(t, \infty)$, $t > 0$. Then*

$$\lim_{n \rightarrow \infty} \frac{m_\Lambda(Z_n)}{n} = 1 \quad (10)$$

for almost all with respect to \varkappa_Λ vectors $Z \in C$.

Note that we may rewrite (10) as

$$\lim_{z \rightarrow 0} \frac{m_\Lambda(z)}{\#\{i : Z_i > y\}} = 1. \quad (11)$$

In other words, if η is a random discrete measure obeying the law P_Λ , then the number of charges of η which are greater than z has the same asymptotics when $z \rightarrow 0$ as the tail $\Lambda(z, \infty)$ of the Lévy measure Λ .

Denote by $D_1^+ \subset D^+$ the simplex of all normalized atomic measures. Then $D^+ = D_1^+ \times [0, \infty)$, i.e. each $\eta \in D^+$ can be represented as

$$\eta = (\eta/\eta(X), \eta(X)). \quad (12)$$

The second coordinate is the total charge of the measure η . It follows from the definition of a Lévy process that $\eta(X)$ obeys the infinite divisible law F_Λ corresponding to the Lévy measure Λ . The first coordinate is called the *normalization* of the measure η . Note that, in general, the law of a Lévy process is not a product measure in this decomposition (see Lemmas 2 and 3 below).

Using this decomposition, consider a map $T' : D^+ \rightarrow \mathbb{R}_+ \times \Sigma \times X^\infty$, where $\Sigma = \{y = (y_1, y_2, \dots) : y_1 \geq y_2 \geq \dots \geq 0, y_1 + y_2 + \dots = 1\}$ is the infinite-dimensional simplex, and

$$T'\eta = (\eta(X), (Q_1/\eta(X), Q_2/\eta(X), \dots), (X_1, X_2, \dots)), \quad \text{if } \eta = \sum Q_i \delta_{X_i}.$$

Definition 4 *The normalized sequence of charges $Q_1/\eta(X), Q_2/\eta(X), \dots$ is called the simplicial part of the process and its distribution σ_Λ on Σ is called the simplicial part of the law P_Λ .*

4 The gamma process and Poisson–Dirichlet distributions

Let (X, ν) be a standard Borel space with a non-atomic finite non-negative measure ν , and let $\nu(X) = \theta$ be the total charge of ν .

Definition 5 *The gamma process with scale parameter $\lambda > 0$ on the space (X, ν) is a Lévy process on (X, ν) corresponding to the Lévy measure with density $d\Lambda_\Gamma^\lambda(z) = z^{-1}e^{-\lambda z}dz$, $z > 0$.*

As in general case, instead of considering a non-normalized parameter measure ν , we may take its normalization $\bar{\nu} = \nu/\nu(X)$ and the Lévy measure with density $\theta z^{-1}e^{-\lambda z}dz$. But in this particular case it is often more convenient to use the above definition with a non-normalized measure. In the sequel we shall use both variants without additional mention.

The corresponding infinitely divisible law is the gamma distribution $\mathcal{G}_{\theta, \lambda}$ on \mathbb{R}_+ with shape parameter θ and scale parameter λ , i.e.

$$d\mathcal{G}_{\theta, \lambda} = \frac{\lambda^\theta}{\Gamma(\theta)} t^{\theta-1} e^{-\lambda t} dt, \quad t > 0.$$

Note that λ is a trivial scale parameter. Namely, if η is the gamma process with scale parameter 1, then the gamma process η^λ with scale parameter λ is obtained from η by multiplying by λ , i.e. $\eta^\lambda = \lambda\eta$. Thus *we will consider only gamma processes with scale parameter 1*.

The law $P_\Gamma = P_\Gamma(\nu)$ of the gamma process (called the *gamma measure* on the space (X, ν)) is thus given by the Laplace transform

$$\mathbb{E}_\Gamma \left[\exp \left(- \int_X a(x) d\eta(x) \right) \right] = \exp \left(- \int_X \log(1 + a(x)) d\nu(x) \right), \quad (13)$$

where a is an arbitrary non-negative bounded Borel function on X .

Let $\mathcal{M} = \mathcal{M}(X, \nu)$ be the set of (classes mod 0 of) non-negative measurable functions on the space X with ν -summable logarithm,

$$\mathcal{M} = \left\{ a : X \rightarrow \mathbb{R}_+ : \int_X |\log a(x)| d\nu(x) < \infty \right\}.$$

It follows from the Poisson construction (6) and Campbell's theorem on sums over Poisson processes (see [20, 3.2]) that each function $a \in \mathcal{M}$ correctly

defines a measurable linear functional $\eta \mapsto f_a(\eta) = \int_X a(x) d\eta(x)$ on D , and formula (13) holds for all $a \in \mathcal{M}$.

It is well known that the gamma distribution enjoys the following property. If Y and Z are independent gamma variables with the same scale parameter, then the variables $Y + Z$ and $\frac{Y}{Y+Z}$ are independent. Moreover, a remarkable result of Lukacs [22] (similar to the famous Bernstein's characterization of normal distributions) states that this property is characteristic of the gamma distribution, i.e. if Y and Z are independent non-degenerate positive random variables, and the variables $Y + Z$ and $\frac{Y}{Y+Z}$ are independent, then both Y and Z have gamma distributions with the same scale parameter. In other words, the independence condition may be formulated as follows. Let us describe a point $x = (x_1, x_2)$ in the first quadrant $\mathbb{R}_+ \times \mathbb{R}_+$ by the sum $x_1 + x_2$ of its coordinates and its projection onto the unit 2-simplex (i.e. interval) $\{y = (y_1, y_2) : y_1, y_2 \geq 0, y_1 + y_2 = 1\}$. Then the distribution $\mathcal{G}_{\theta_1, \lambda} \times \mathcal{G}_{\theta_2, \lambda}$ on $\mathbb{R}_+ \times \mathbb{R}_+$ is a product measure in these coordinates. These results imply the corresponding statements for the gamma process which are a key point for many important properties of P_Γ .

Lemma 2 *In representation (12) the gamma measure is a product measure $P_\Gamma = \mathcal{G}_\theta \times \bar{P}_\Gamma$, i.e. the total charge $\gamma(X)$ of the gamma process and the normalized gamma process $\bar{\gamma} = \gamma/\gamma(X)$ are independent. The distribution of the total charge is the gamma distribution $\mathcal{G}_{\theta, 1}$ with shape parameter θ and scale parameter 1.*

Lemma 3 *If the law P_Λ of some Lévy process is a product measure in representation (12), then P_Λ is a gamma process, i.e. $d\Lambda(z) = z^{-1}e^{-\lambda z}dz$, $z > 0$, for some $\lambda > 0$.*

Definition 6 ([18]) *The simplicial part of the gamma measure $P_\Gamma(\nu)$ with $\nu(X) = \theta$ is called the Poisson–Dirichlet distribution with parameter θ and denoted by $\text{PD}(\theta)$.*

The above definition is just one of many other possible definitions of the Poisson–Dirichlet distributions. These distributions arise in many fields of pure and applied mathematics. They play an important role in statistics because of their connection with Dirichlet distributions [18] and Dirichlet random measures [10]. In number theory Poisson–Dirichlet measures arise

in the problem of distribution of prime divisors of a random integer [3, 30]. There are many asymptotic combinatorial problems leading to the measures $\text{PD}(\theta)$, such as the distribution of the cycle lengths of a random permutation [33] or the distribution of the degrees of the irreducible factors of a random monic polynomial over a finite field [13], etc. The Poisson–Dirichlet distributions also play an important role in applications to population genetics, ecology and physics. A (non-complete) survey of different aspects of Poisson–Dirichlet measures can be found in [9].

It follows from Lemma 2 that $T'P_\Gamma = \mathcal{G}_\theta \times \text{PD}(\theta) \times \bar{\nu}^\infty$, i.e. the conic part of the gamma measure is a product measure $\mathcal{G}_{\theta,1} \times \text{PD}(\theta)$. We call this measure the *conic Poisson–Dirichlet distribution* with parameter θ and denote it by $\text{CPD}(\theta)$.

Note that in case of the gamma process $m(t) = \theta \int_t^\infty s^{-1} e^{-s} ds \sim -\theta \log t$. Thus, by Proposition 1,

$$\lim_{n \rightarrow \infty} \frac{\log Z_n}{n} = -\frac{1}{\theta} \quad (14)$$

almost surely with respect to $\text{CPD}(\theta)$. It follows that the same asymptotics holds for $\text{PD}(\theta)$, i.e. $\lim_{n \rightarrow \infty} \frac{\log Y_n}{n} = -\frac{1}{\theta}$ for almost all vectors $Y \in \Sigma$ with respect to $\text{PD}(\theta)$. In particular, we see that the measures $\text{PD}(\theta)$ (as well as $\text{CPD}(\theta)$) are mutually singular for different θ .

Many properties of ordinary Poisson–Dirichlet distributions have their natural analogues for conic distributions. For example, it is well known that the measure $\text{PD}(\theta)$ may be obtained by the following *stick breaking process*. Let Y_1 be a random variable on the interval $[0, 1]$ obeying the law $\theta(1-t)^{\theta-1} dt$, $t \in [0, 1]$. If we have already constructed variables Y_1, \dots, Y_n , then Y_{n+1} has the same distribution scaled on the interval $[Y_n, 1]$. Thus we obtain a random sequence $0 = Y_0 < Y_1 < Y_2 < \dots < 1$. Let $Z_k = Y_k - Y_{k-1}$, $k = 1, 2, \dots$. The Poisson–Dirichlet measure $\text{PD}(\theta)$ is the distribution of the order statistics $Z_{(1)} \geq Z_{(2)} \geq \dots$ of the sequence Z_1, Z_2, \dots . It follows that the conic Poisson–Dirichlet measure $\text{CPD}(\theta)$ may be obtained by the randomized version of this procedure. Namely, for the first step we choose the random length L of the interval with gamma distribution $\mathcal{G}_{\theta,1}$, and then proceed as before starting with the interval $[0, L]$.

As shown in [25], the Poisson–Dirichlet measures $\text{PD}(\theta)$ can be naturally included into a two-parameter family $\text{PD}(\alpha, \theta)$ of distributions on the simplex Σ . This family is obtained by the following non-stationary version of the stick breaking process. Let Y_1 be a random variable on the interval $[0, 1]$ obeying

the beta distribution $B(1-\alpha, \theta+\alpha)$. If we have already constructed variables Y_1, \dots, Y_n , then Y_{n+1} has the beta distribution $B(1-\alpha, \theta+(n+1)\alpha)$ scaled on the interval $[Y_n, 1]$. Thus we obtain a random sequence $0 = Y_0 < Y_1 < Y_2 < \dots < 1$. Let $Z_k = Y_k - Y_{k-1}$, $k = 1, 2, \dots$. The two-parameter Poisson–Dirichlet measure $\text{PD}(\alpha, \theta)$ is the distribution of the order statistics $Z_{(1)} \geq Z_{(2)} \geq \dots$ of the sequence Z_1, Z_2, \dots .

The range of admissible parameters is the union of the sets

$$\{(\alpha, \theta) : 0 \leq \alpha < 1, \theta > -\alpha\} \quad \text{and} \quad \{(\alpha, -m\alpha) : \alpha < 0, m \in \mathbb{N}\}.$$

The first case $\alpha \in (0, 1)$ is the most interesting, the second one being a sort of degenerate case. The ordinary Poisson–Dirichlet distributions $\text{PD}(\theta)$ correspond to $\alpha = 0$, i.e. $\text{PD}(\theta) = \text{PD}(0, \theta)$. See [25] for various properties of the measures $\text{PD}(\alpha, \theta)$. In particular, the distributions $\text{PD}(\alpha, \theta)$ with fixed $\alpha \neq 0$ and different θ are absolutely continuous (unlike the case $\alpha = 0$). We discuss some questions related to the two-parameter Poisson–Dirichlet distributions in Sections 8 and 12.

5 Multiplicative quasi-invariance of the gamma process

As was mentioned above, the law P_Λ of each Lévy process is invariant under all ν -preserving transformations of the space (X, ν) which act on D by substituting the coordinates. However, the gamma measure P_Γ enjoys additional invariance properties. We present now a large group of linear transformations of the space D (preserving the cone D^+) for which P_Γ is a quasi-invariant measure.

Consider the above defined class \mathcal{M} of non-negative functions on X with ν -summable logarithm. Each function $a \in \mathcal{M}$ defines not only a linear functional f_a on D but also a multiplier $M_a : D \rightarrow D$ by $(M_a \eta)(x) = a(x)\eta(x)$, that is $M_a \eta = \sum a(x_i) z_i \delta_{x_i}$ for $\eta = \sum z_i \delta_{x_i}$. Note that the set \mathcal{M} is a commutative group with respect to pointwise multiplication of functions, and M_a is a group action of \mathcal{M} . Denote by \tilde{a} the function $\tilde{a}(x) = (1/a(x)) - 1$.

The following property of the gamma process was first discovered in [32, 12] in quite different terms; it plays an important role in the representation theory of the current group $\text{SL}(2, F)$, where F is the space of functions on a manifold.

Theorem 3 For each $a \in \mathcal{M}$, the measure P_Γ is quasi-invariant under M_a , and the corresponding density is given by the following formula,

$$\frac{d(M_a P_\Gamma)}{dP_\Gamma}(\eta) = \exp \left(- \int_X \log a(x) d\nu(x) \right) \cdot \exp \left(- \int_X \tilde{a}(x) d\eta(x) \right). \quad (15)$$

Proof. Fix $a \in \mathcal{M}$ and let $\xi = L_a \eta$. Consider an arbitrary function $b \in \mathcal{M}$. Then $\int_X b(x) d\xi(x) = \int_X a(x) b(x) d\eta(x)$. Thus, in view of (13), the Laplace transform $\mathbb{E} \left[\exp \left(- \int_X b(x) d\xi(x) \right) \right]$ equals

$$\begin{aligned} \mathbb{E} \left[\exp \left(- \int_X a(x) b(x) d\eta(x) \right) \right] &= \exp \left(- \int_X \log (1 + a(x) b(x)) d\nu(x) \right) = \\ &= \exp \left(- \int_X \log a(x) d\nu(x) \right) \cdot \exp \left(- \int_X \log \left(\frac{1}{a(x)} + b(x) \right) d\nu(x) \right). \end{aligned}$$

Using (13) once more, we may consider the last factor as the Laplace transform of P_Γ calculated on the function $(\frac{1}{a(x)} - 1) + b(x) = \tilde{a}(x) + b(x)$. Denote $I(a) = \int_X \log a(x) d\nu(x)$. Then we have

$$\begin{aligned} \mathbb{E} \left[\exp \left(- \int_X b(x) d\xi(x) \right) \right] &= I(a) \cdot \mathbb{E} \left[\exp \left(- \int_X (\tilde{a}(x) + b(x)) d\eta(x) \right) \right] = \\ &= \mathbb{E} \left[I(a) \cdot \exp \left(- \int_X \tilde{a}(x) d\eta(x) \right) \cdot \exp \left(- \int_X b(x) d\eta(x) \right) \right], \end{aligned}$$

and Theorem 3 follows. \square

In particular, if we consider multiplication by constant $c > 0$, then the corresponding density depends only on the total charge, namely

$$\frac{d(M_c P_\Gamma)}{dP_\Gamma}(\eta) = \frac{1}{c^\theta} \cdot \exp \left(\left(1 - \frac{1}{c} \right) \eta(X) \right). \quad (16)$$

Theorem 4 The action of the group \mathcal{M} on the space (D^+, P_Γ) is ergodic.

Proof. Let $G : D^+ \rightarrow \mathbb{R}$ be a P_Γ -measurable functional on D^+ which is invariant under all M_a i.e. $G(M_a \eta) = G(\eta)$ a.e. with respect to P_Γ . Consider an arbitrary Borel function $k : \mathbb{R} \rightarrow \mathbb{R}$. Then for each $a \in \mathcal{M}$

$$\begin{aligned} \mathbb{E}[k(G(\eta))] &= \mathbb{E}[k(G(M_a \eta))] = \\ &= \mathbb{E} \left[k(G(\eta)) \cdot \exp \left(- \int_X \tilde{a}(x) d\eta(x) \right) \right] \cdot \exp \left(- \int_X \log a(x) d\nu(x) \right), \end{aligned}$$

where \mathbb{E} denotes the expectation with respect to P_Γ . But in view of (13) the last factor equals

$$\left(\mathbb{E} \left[\exp \left(- \int_X \tilde{a}(x) d\eta(x) \right) \right] \right)^{-1} = (\mathbb{E} [\exp (-f_{\tilde{a}}(\eta))])^{-1},$$

hence we have

$$\mathbb{E} [k(G(\eta)) \exp (-f_{\tilde{a}}(\eta))] = \mathbb{E} [k(G(\eta))] \cdot \mathbb{E} [\exp (-f_{\tilde{a}}(\eta))].$$

Thus G is independent of every functional f_a , and Theorem 4 follows. \square

It is natural to ask whether the quasi-invariance property stated in Theorem 3 is characteristic of the gamma process. If we fix the density, then the answer is positive ([12]), i.e. the gamma measure is the only measure on D^+ satisfying (15). On the other hand, for a smaller group of multiplications, the answer may be negative, as the following example shows. Let us call a process *quasi-multiplicative* if its law is invariant under all transformations M_a with *constant* $a > 0$. For simplicity, consider *subordinators* (i.e. Lévy processes for $X = \mathbb{R}_+$ and ν equal to the Lebesgue measure).

Proposition 2 ([34]) *Let η be a subordinator with Lévy measure $\Lambda(dx) = k(x)dx$, where $k(x) > 0$, and denote $g(x) = xk(x)$. Then η is quasi-multiplicative if and only if for all $a > 0$*

$$\int_0^1 \left(\sqrt{g(x/a)} - \sqrt{g(x)} \right)^2 \frac{dx}{x} < \infty. \quad (17)$$

It is shown in [34] that for each $m < 1/2$ any function $k_m(x)$ that satisfies

$$k_m(x) = \frac{1}{x} \left(\log \frac{1}{x} \right)^{2m} \quad \text{for } x < 1/2 \quad \text{and} \quad \int_{1/2}^\infty k_m(x) dx < \infty \quad (18)$$

provides an example of a quasi-multiplicative subordinator that is not equivalent to any scaled gamma process. It is clear that if η is quasi-multiplicative, then its law is quasi-invariant under all step functions with finitely many steps, but the quasi-invariance under the whole group \mathcal{M} does not take place in this example. It is not known if there exists a measure different from P_Γ which has this property.

Computation of the law of hitting time of the drifted gamma process is coherent with the quasi-invariance of the gamma process. This fact is also parallel to some property of Brownian motion and will be considered in more details elsewhere.

6 Quasi-invariance of the Poisson–Dirichlet distributions

Let $a \in \mathcal{M}$. According to the general theory of polymorphisms (see [29]), the transformation M_a induces a Markovian operator R_a on the cone C . Namely, let $z = (z_1, z_2, \dots) \in C$. Consider the conditional distribution P_Γ^z of the gamma process on (X, ν) , given the conic part equal to z . Then the random image of the point z under R_a is the conic part of the process $M_a \eta$, where η obeys the law P_Γ^z . It follows from Theorem 1 that

$$R_a z = V(a(X_1)z_1, a(X_2)z_2, \dots),$$

where (X_1, X_2, \dots) is a sequence of i.i.d. random variables on X with common distribution ν , and V denotes a map that arranges the coordinates in non-increasing order.

In a similar way, the transformation M_a induces a Markovian operator S_a on the simplex Σ ,

$$S_a y = V\left(\frac{a(X_1)y_1}{\sigma}, \frac{a(X_2)y_2}{\sigma}, \dots\right),$$

where the sequence (X_1, X_2, \dots) is as before, and $\sigma = a(X_1)y_1 + a(X_2)y_2 + \dots$.

Note that the definitions of the operators S_a and R_a depend only on the distribution of the function a . Thus we may assume that $X = [0, 1]$ and $\nu = \theta \lambda$, where λ is the Lebesgue measure on the interval.

Theorems 3, 4 immediately imply

Theorem 5 1) *The Poisson–Dirichlet distribution $\text{PD}(\theta)$ is quasi-invariant under the Markovian operator S_a for all $a \in \mathcal{M}$, and*

$$\frac{dS_a \text{PD}(\theta)}{d\text{PD}(\theta)}(y) = \exp\left(-\theta \int_0^1 \log a(s) ds\right) \cdot \int_0^\infty \frac{\sigma^{\theta-1}}{\Gamma(\theta)} \left(\prod_{i=1}^\infty L_{1/a}(\sigma y_i)\right) d\sigma,$$

where $L_{1/a}(\cdot)$ is the Laplace transform of the distribution of the function $1/a(t)$ with respect to the uniform distribution on the interval $[0, 1]$.

2) *The Poisson–Dirichlet distribution $\text{PD}(\theta)$ is ergodic with respect to $\{S_a\}_{a \in \mathcal{M}}$.*

7 Quasi-Lebesgue measure and a representation of the current group

In this section we define, following [32, 12], a σ -finite measure on D^+ which is equivalent to P_Γ and invariant under a subgroup \mathcal{M}_0 of \mathcal{M} consisting of all functions $a \in \mathcal{M}$ such that $\int_X \log a(x) d\nu(x) = 0$.

Definition 7 Consider a σ -finite measure \tilde{P}_Γ on D^+ defined by

$$\frac{d\tilde{P}_\Gamma}{dP_\Gamma}(\eta) = \exp(\eta(X)). \quad (19)$$

It is called the quasi-Lebesgue measure.

Theorem 3 implies

Theorem 6 For each $a \in \mathcal{M}$, the quasi-Lebesgue measure \tilde{P}_Γ is quasi-invariant under M_a with a constant density

$$\frac{dM_a(\tilde{P}_\Gamma)}{d\tilde{P}_\Gamma} = \exp\left(-\int_X \log a(x) d\nu(x)\right).$$

Corollary 1 If $\int_X \log a(x) d\nu(x) = 0$, then the quasi-Lebesgue measure \tilde{P}_Γ is invariant with respect to M_a .

We see that the measure \tilde{P}_Γ is invariant with respect to an infinite-dimensional multiplicative group whose action generalizes the action of the group of diagonal matrices with determinant 1 in a finite-dimensional vector space. Thus we may consider the measure \tilde{P}_Γ as an infinite-dimensional analogue of the Lebesgue measure. This property was much used in [32, 12] for the representation theory of the group $\mathrm{SL}(2, F)$.

Let us consider the group of triangular matrices of order 2

$$T_{a,b} = \begin{pmatrix} a(\cdot) & b(\cdot) \\ 0 & a(\cdot)^{-1} \end{pmatrix}$$

with $b, \log a \in L^1(X, m)$ for some measurable space (X, m) (a current group in the terminology of physicists).

Theorem 7 *The formula*

$$U(T_{a,b})F(\eta) = \exp \left(\int_X \log a(x) d\nu(x) + i \int_X a(x)b(x) d\eta(x) \right) F(M_a\eta).$$

defines a unitary irreducible representation of this group in the space $L^2(\tilde{P}_\Gamma)$.

Proof. The representation is correctly defined and its unitarity follows from the invariance property of \tilde{P}_Γ . The irreducibility follows from the ergodicity of the action of the group \mathcal{M} of multipliers. \square

Remarks. 1. This representation may be extended to the group $\mathrm{SL}(2, F)$, for this we need to define only one operator, namely, the image of the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

2. This representation of $\mathrm{SL}(2, F)$ was firstly introduced in [31] in a completely different way, but later [32, 12] was interpreted with the space $L^2(\tilde{P}_\Gamma)$.

3. This representation is a continual analogue of the classical representations of the group $\mathrm{SL}(2, \mathbb{R})$ in $L^2(\mathbb{R}_+)$.

Note that passing from P_Γ to \tilde{P}_Γ we do not change the conditional measures P_Γ^s of P_Γ , given the conic part equal to s , and modify only the factor measure on the half-line, i.e. $T'\tilde{P}_\Gamma = m_\theta \times \mathrm{PD}(\theta) \times \bar{\nu}^\infty$ and $T\tilde{P}_\Gamma = \widetilde{\mathrm{PD}}(\theta) \times \bar{\nu}$, where m_θ has density $t^{\theta-1} / \Gamma(\theta)$, $t > 0$ (in particular, m_1 is just the Lebesgue measure on the half-line), and $\widetilde{\mathrm{PD}}(\theta) = m_\theta \times \mathrm{PD}(\theta)$.

Recall that the transformation M_a induces a Markovian operator R_a on the cone C (see Section 6).

Corollary 2 *The σ -finite measure $\widetilde{\mathrm{PD}}(\theta)$ on the cone C is invariant under the Markovian operator R_a for all $a \in \mathcal{M}$.*

8 Stable processes and general Poisson–Dirichlet measures

Definition 8 *Let $\alpha \in (0, 1)$. The standard α -stable process on the space (X, ν) is a Lévy process with Lévy measure*

$$d\Lambda_\alpha = \frac{c\alpha}{\Gamma(1-\alpha)} s^{-\alpha-1} ds, \quad s > 0, \quad (20)$$

where $c > 0$ is an arbitrary fixed positive number.

The corresponding infinitely divisible law, i.e. the distribution of the sum of charges with respect to P_α , is the α -stable law F_α on \mathbb{R}_+ .

Denote by P_α the law of the α -stable process. The Laplace transform of P_α equals

$$\mathbb{E}_\alpha \left[\exp \left(- \int_X a(x) d\eta(x) \right) \right] = \exp \left(-c \int_X a(x)^\alpha d\nu(x) \right), \quad (21)$$

for an arbitrary measurable function $a : X \rightarrow \mathbb{R}_+$ with $\int_X a(x)^\alpha d\nu(x) < \infty$.

Proposition 3 ([25]) *The simplicial part of an α -stable process with $\alpha \in (0, 1)$ is the Poisson–Dirichlet distribution $\text{PD}(\alpha, 0)$.*

Proposition 4 *The conic part of the law of the α -stable process is concentrated on the set*

$$\left\{ z \in C : \lim_{n \rightarrow \infty} z_n n^{1/\alpha} = \left(\frac{c}{\Gamma(1-\alpha)} \right)^{1/\alpha} \right\}. \quad (22)$$

Proof. In case of an α -stable process, we have $m(t) = \frac{c}{\Gamma(1-\alpha)} t^{-\alpha}$, and (22) follows immediately from Proposition 1. \square

For $s > 0$, denote by \varkappa_α^s the conditional distribution of the conic part \varkappa_α of the stable process on the simplex $\Sigma_s = \{z = (z_1, z_2, \dots) : \sum z_i = s\}$ of monotone sequences with sum s (i.e. the conic part of the conditional distribution of the law P_α on the set $D_s = \{\eta = \sum z_i \delta_{x_i} \in D^+ : \sum z_i = s\}$ of positive discrete measures with total charge s).

Corollary 3 *The homothetic projection of the conditional measure \varkappa_α^s on the unit simplex Σ is concentrated on the set*

$$\left\{ y \in \Sigma : \lim_{n \rightarrow \infty} y_n n^{1/\alpha} = \frac{1}{s} \left(\frac{c}{\Gamma(1-\alpha)} \right)^{1/\alpha} \right\} \quad (23)$$

Corollary 4 *The homothetic projections of the measures \varkappa_α^s and \varkappa_α^t on the unit simplex Σ are singular for all pairs $s, t > 0$, $s \neq t$. Thus the distribution $\text{PD}(\alpha, 0)$ is a continual sum of a family of singular distributions.*

The following statement shows how one may recover the conic part of the stable process starting with its simplicial part $\text{PD}(\alpha, 0)$.

Corollary 5 ([25]) *Let the vector $Q = (Q_1, Q_2, \dots) \in \Sigma$ have the distribution $\text{PD}(\alpha, 0)$. The limit $L(Q) = \lim_{n \rightarrow \infty} n^{1/\alpha} Q_n$ exists almost surely. Let $S(Q) = \frac{1}{L(Q)} \cdot \frac{c}{\Gamma(1-\alpha)}$. Then the α -stable process η on the space (X, ν) may be represented as*

$$\eta = S(Q) \sum_{i=1}^{\infty} Q_i \delta_{X_i},$$

where Q obeys $\text{PD}(\alpha, 0)$ and X_1, X_2, \dots is a sequence of i.i.d. variables on X with common distribution ν .

The Poisson–Dirichlet distribution $\text{PD}(\alpha, \theta)$ with $\alpha, \theta \neq 0$ is not the simplicial part of any Lévy process. However, one may obtain it as the simplicial part of the process that has density with respect to a stable process. Namely, let $\theta > -\alpha$ and consider the law $P_{\alpha, \theta}$ on D which has the density

$$\frac{dP_{\alpha, \theta}}{dP_{\alpha}}(\eta) = \frac{c_{\alpha, \theta}}{\eta(X)^{\theta}} \quad (24)$$

with respect to the α -stable law P_{α} . Here $c_{\alpha, \theta} = c^{\theta/\alpha} \frac{\Gamma(\theta+1)}{\Gamma(\theta/\alpha+1)}$ is a normalizing constant.

Proposition 5 ([25]) *The simplicial part of the law $P_{\alpha, \theta}$ is the Poisson–Dirichlet distribution $\text{PD}(\alpha, \theta)$.*

9 The gamma measure as a weak limit of laws of α -stable processes when α tends to zero

The purpose of this section is to show that it is natural to consider the gamma process as a weak limit of the α -stable processes when $\alpha \rightarrow 0$. We present several settings of this statement.

Let $k > 0$ and consider the measure $P_{\alpha, k}$ on D given by

$$\frac{dP_{\alpha, k}}{dP_{\alpha}}(\eta) = \frac{\exp(-\gamma \cdot \eta(X))}{\mathbb{E}_{\alpha}[\exp(-\gamma \cdot \eta(X))]} = e^{\gamma^{\alpha}} \cdot e^{-\gamma \cdot \eta(X)}, \quad (25)$$

where $\gamma = \frac{k}{\alpha^{1/\alpha}}$. Denote by $\hat{P}_{\alpha, k}$ the law of the process $\gamma \cdot \eta$, where η obeys $P_{\alpha, k}$. The following theorem was formulated in [34].

Theorem 8 ([34]) *The measures $\hat{P}_{\alpha,k}$ converge weakly to P_Γ when $\alpha \rightarrow 0$.*

Proof. It follows from (25), (21) that the Laplace transform of the measure $\hat{P}_{\alpha,k}$ equals

$$\mathbb{E}_{\hat{P}_{\alpha,k}} \exp(-f_a(\eta)) = \exp \left(-\gamma^\alpha \int_X ((a(x) + 1)^\alpha - 1) d\nu(x) \right).$$

But

$$\gamma^\alpha ((a(x) + 1)^\alpha - 1) = \frac{k^\alpha}{\alpha} (\alpha \log(a(x) + 1) + o(\alpha)) \rightarrow \log(a(x) + 1),$$

as $\alpha \rightarrow 0$, hence

$$\mathbb{E}_{\hat{P}_{\alpha,k}} \exp(-f_a(\eta)) \rightarrow \exp \left(- \int_X \log(a(x) + 1) d\nu(x) \right) = \mathbb{E}_{P_\Gamma} \exp(-f_a(\eta)),$$

and Theorem 7 follows. \square

This important result is a key point of the construction of two-parameter Poisson–Dirichlet distributions $\text{PD}(\alpha, \theta)$ (see Section 4). In particular, one obtains the following corollary.

Corollary 6 ([25]) *For a fixed $\theta \neq 0$, the distributions $\text{PD}(\alpha, \theta)$ converge to $\text{PD}(0, \theta) = \text{PD}(\theta)$ when $\alpha \rightarrow 0$.*

Corollary 7 *Let $\tilde{P}_{\alpha,k}$ be a measure with constant density e^{γ^α} with respect to the α -stable law P_α . Denote by $\check{P}_{\alpha,k}$ the law of the process $\gamma \cdot \eta$, where η obeys $\tilde{P}_{\alpha,k}$. Then the measures $\check{P}_{\alpha,k}$ converge weakly to the quasi-Lebesgue measure \tilde{P}_Γ .*

That the quasi-Lebesgue measure is a kind of the limit case of stable processes was suggested in [21]. See Section 10 for another understanding of this statement.

There is the following useful way to formalize the transition to a σ -finite limit. Consider the convolution of n copies of the measures $P_{1/n}$ and multiply it by a function of n . Then the limit in n will be the σ -finite measure under consideration. In terms of the Laplace transform this procedure is equivalent to the following relation:

$$\lim_{n \rightarrow \infty} \exp(-n(x^{1/n} - 1)) = \frac{d}{d\alpha} \exp(-x^\alpha) |_{\alpha=0} = \frac{1}{x}.$$

Differentiating by α at the point $\alpha = 0$ is just a “commutative” analogue of the main technique used in [31] for constructing so-called canonical states and the representation theory of semi-simple currents for the group $\text{SL}(2, R)$.

10 Equivalent definition of stable laws, σ -finite stable measures and zero-stable laws

In this section we give another definition of stable laws which is valid for σ -finite measures. We describe a piece of theory (to be exposed in details elsewhere) of σ -finite stable measures showing that it is natural to consider the Lebesgue measure as a zero-stable law.

Let F be a distribution on \mathbb{R} . Consider the distribution $F \times F$ on $\mathbb{R} \times \mathbb{R}$. Let $\|\cdot\|_\alpha$ be the α -norm in the space $(\mathbb{R} \times \mathbb{R})^*$ of linear functionals on $\mathbb{R} \times \mathbb{R}$, i.e. if $f(x_1, x_2) = a_1x_1 + a_2x_2$, then $\|f\|_\alpha = (|a_1|^\alpha + |a_2|^\alpha)^{1/\alpha}$. Let us consider only stable laws depending on one parameter $\alpha \in (0, 2]$. Then the ordinary definition of an α -stable law on \mathbb{R} is equivalent to the following one.

Definition 9 *The law F is α -stable, if the following condition holds. If two linear functionals f_1 and f_2 on $\mathbb{R} \times \mathbb{R}$ have the same α -norm, i.e. $\|f_1\|_\alpha = \|f_2\|_\alpha$, then f_1 and f_2 have the same distribution with respect to $F \times F$.*

Note that in case of linear functionals f_1 and f_2 the equality of distributions is equivalent to the existence of a $F \times F$ -preserving transformation $L : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$ such that $f_2 = f_1 \circ L$. Thus we obtain the following definition of a stable law which applies to σ -finite measures.

Definition 10 *The measure F (may be σ -finite) is called α -stable, if the following condition holds. If two linear functionals f_1 and f_2 on $\mathbb{R} \times \mathbb{R}$ have the same α -norm, i.e. $\|f_1\|_\alpha = \|f_2\|_\alpha$, then there exists a $F \times F$ -preserving transformation $L : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$ such that $f_2 = f_1 \circ L$.*

Let $a_1a_2 \neq 0$. If $\alpha \rightarrow 0$, then

$$\begin{aligned} 2^{-1/\alpha} \|f\|_\alpha &= \left(\frac{|a_1|^\alpha + |a_2|^\alpha}{2} \right)^{1/\alpha} = \left(\frac{1}{2} |a_1|^\alpha + \frac{1}{2} |a_2|^\alpha \right)^{1/\alpha} = \\ &= \left(\frac{1}{2} + \alpha \log |a_1| + O(\alpha^2) + \frac{1}{2} + \alpha \log |a_2| + O(\alpha^2) \right)^{1/\alpha} = \\ &= (1 + \alpha \log |a_1a_2| + O(\alpha^2))^{1/\alpha} \rightarrow \log |a_1a_2|, \end{aligned}$$

thus it is natural to consider the quasi-norm $\|f\|_0 = |a_1a_2|$ as a natural limit of α -norms when α tends to 0. We obtain the following definition of a zero-stable measure on \mathbb{R} .

Definition 11 *The measure F on \mathbb{R} is called zero-stable, if for each two linear functionals f_1 and f_2 on $\mathbb{R} \times \mathbb{R}$ with $\|f_1\|_0 = \|f_2\|_0 < \infty$, there exists a transformation $L : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$ preserving $F \times F$ such that $f_2 = f_1 \circ L$.*

Remark. Note that in case $\alpha = 2$, corresponding to the normal law F , the $F \times F$ -preserving transformation L , which connects functionals of equal 2-norm, is a rotation. In the “opposite” case $\alpha = 0$ the corresponding transformation is hyperbolic, of the form $(a_1, a_2) \mapsto (ca_1, \frac{a_2}{c})$. In both cases we have a linear mapping. Unlike these two extreme cases, in intermediate cases $\alpha \in (0, 2)$ the transformation L is not linear.

Proposition 6 *For any $\beta < 1$, the measure on \mathbb{R} with density $|x|^{-\beta}dx$ is zero-stable. In particular, the Lebesgue measure on \mathbb{R} is zero-stable.*

Proof. Easy calculation. □

Now we may construct a theory of σ -finite stable processes on arbitrary spaces. Definition 8 of an α -stable process on the space X is equivalent to the following one. We recall that each bounded Borel function a on X defines a linear functional f_a on the space D of finite discrete measures on X by $f_a(\eta) = \int_X a(x)d\eta(x)$. Let $\|\cdot\|_\alpha$ denote the α -norm $\|a\|_\alpha = (\int |a(x)|^\alpha d\nu(x))^{1/\alpha}$, and let $\|a\|_0 = \exp(\int_X \log |a(x)| d\nu(x))$.

Definition 12 *The law P_α on D is α -stable, where $\alpha \in [0, 2]$, if for each two linear functionals f_{a_1} and f_{a_2} with $\|a_1\|_\alpha = \|a_2\|_\alpha < \infty$, there exists a transformation $L : D \rightarrow D$ preserving the law P_α such that $f_{a_2} = f_{a_1} \circ L$.*

It is easy to check that the quasi-Lebesgue measure \tilde{P}_Γ satisfies this condition. Indeed, if $\|a_1\|_0 = \|a_2\|_0$, then $\int_X \log(a_2/a_1)(x)d\nu(x) = 0$. Hence the multiplier M_{a_2/a_1} preserves the measure \tilde{P}_Γ by Corollary 1, and it is obvious that $f_{a_2} = f_{a_1} \circ M_{a_2/a_1}$. Thus we obtain the following proposition.

Proposition 7 *The quasi-Lebesgue measure \tilde{P}_Γ is zero-stable.*

Note that formulae (13) and (19) imply that for all $a \in \mathcal{M}$

$$\begin{aligned} \mathbb{E}_{\tilde{P}_\Gamma} \left[\exp \left(- \int_X a(x)d\eta(x) \right) \right] &= \exp \left(- \int_X \log a(x)d\nu(x) \right) \\ &= \exp \left(\int_X \log \phi(a(x))d\nu(x) \right), \end{aligned}$$

where $\phi(t) = 1/t$ is the Laplace transform of the Lebesgue measure on \mathbb{R}_+ which is zero-stable. Comparing with (5), we see that we could define a zero-stable process as a Lévy process with Lévy measure corresponding to a zero-stable law.

11 Distributions of linear functionals of the gamma processes and the Markov–Krein transform

In this section we show that the Markov–Krein identity known in the context of Dirichlet processes may be interpreted as a formula relating the distribution of a linear functional with respect to the gamma process and the distribution of the same functional with respect to the normalized gamma process. This interpretation allows to prove it immediately, using only a formula for the Laplace transform of the gamma process.

Given a function $a \in \mathcal{M}$, denote by μ_a the distribution of the linear functional $\eta \mapsto f_a(\eta) = \int_X a(x) d\eta(x)$ on D with respect to the law \bar{P}_Γ of the normalized gamma process, and let ν_a be the distribution of the function a with respect to the (normalized) parameter measure ν . The following property is characteristic for the gamma process.

Theorem 9 *The measures μ_a and ν_a are related by the following integral identity,*

$$\int_{\mathbb{R}} \frac{1}{(1+zu)^\theta} d\mu_a(u) = \exp \left(- \int_X \log(1+zu)^\theta d\nu_a(u) \right). \quad (26)$$

This formula was first obtained in [5] in the context of Dirichlet processes by hard analytic arguments (see also simpler proofs in [7] and [16]). But the relation with the gamma process, which is a key point of our simple proof, has been overlooked. Note that the left-hand side of identity (26) is the generalized Cauchy–Stieltjes transform of the distribution μ_a . It is natural to call the right-hand side the multiplicative version of the generalized Cauchy–Stieltjes transform of the distribution ν_a . In view of (13), it is equal to the Laplace transform of the gamma measure P_Γ calculated on the function a . Thus one may regard formula (26) as relating an integral transform (namely, the Cauchy–Stieltjes transform) of the distribution μ_a of the functional f_a

with respect to the normalized gamma process and an integral transform (namely, the Laplace transform) of its distribution with respect to the non-normalized gamma process.

In case of $\theta = 1$, the identity (26) means that the distribution μ_a is the *Markov–Krein transform* of the measure ν_a . This transform arises in many contexts, such as the Markov moment problem, continued fractions theory, exponential representations of functions of negative imaginary type, the Plancherel growth of Young diagrams, etc. (see [15] for a detailed survey).

Proof. Using (13), Lemma 2 and the Fubini theorem we obtain that the right-hand side of (26) equals

$$\begin{aligned} & \exp \left(- \int_X \log(1 + za(x)) d\nu(x) \right) = \mathbb{E}_{P_T} \left[\exp \left(-z \int_X a(x) d\gamma(x) \right) \right] \\ &= \mathbb{E}_{P_T} \left[\exp \left(-z\gamma(X) \int_X a(x) d\bar{\gamma}(x) \right) \right] \\ &= \mathbb{E}_{P_{\bar{T}}} \left[\frac{1}{\Gamma(\theta)} \int_0^\infty t^{\theta-1} \exp \left(-t - zt \int_X a(x) d\bar{\gamma}(x) \right) \right] \\ &= \mathbb{E}_{P_{\bar{T}}} \left[\frac{1}{(1 + z \int_X a(x) d\bar{\gamma}(x))^\theta} \right], \end{aligned}$$

and Theorem follows. \square

Remarks. 1. According to a personal communication of P. Diaconis, the idea of proving formula (26) using the Laplace transform formula for the gamma process was used by F. Huffer in case of *discrete* parameter measure ν (in this case the gamma process is just a sum of independent gamma variables and the normalized gamma process is a random point of a finite-dimensional simplex obeying a Dirichlet distribution). But the fact that this argument works for continuous parameter measures as well, which simplifies the proof in essential manner, seems to have been overlooked.

2. It follows from the known results on the Markov–Krein transform that the distribution μ_a of a linear functional f_a is absolutely continuous (see [5] for an explicit formula for the density).

3. See [16] for similar results on the common distributions of several linear functionals of the Dirichlet process. It is easy to extend the proof of Theorem 9 to obtain these results.

12 The two-parameter generalization of the Markov–Krein formula

In [27] an analogue of the Markov–Krein identity is obtained for the distribution of a linear functional with respect to the generalized Dirichlet process associated with the two-parameter Poisson–Dirichlet distributions $\text{PD}(\alpha, \theta)$. Using the key idea of Section 11, we present here a new proof of this identity based on relation of the two-parameter Poisson–Dirichlet family to the stable processes.

Let $\alpha \in (0, 1)$ and $\theta > -\alpha$. Denote by $\bar{P}_{\alpha, \theta}$ the normalization (in sense of (12)) of the law $P_{\alpha, \theta}$. (Recall that the simplicial part of this law is $\text{PD}(\alpha, \theta)$, see Section 8.) Given an arbitrary measurable function $a : X \rightarrow \mathbb{R}_+$ such that $\int_X a(x)^\alpha d\nu(x) < \infty$, let μ_a be the distribution of the functional f_a with respect to $\bar{P}_{\alpha, \theta}$, and let ν_a be the distribution of the function a with respect to the (normalized) measure $\bar{\nu}$.

Theorem 10 *The measures μ_a and ν_a are related by the following integral identity:*

1) if $\theta \neq 0$,

$$\left(\int_{\mathbb{R}} (1 + zu)^{-\theta} d\mu_a(u) \right)^{-\frac{1}{\theta}} = \left(\int_{\mathbb{R}} (1 + zu)^\alpha d\nu_a(u) \right)^{\frac{1}{\alpha}}; \quad (27)$$

2) if $\theta = 0$,

$$\exp \left(\int_{\mathbb{R}} \log(1 + zu)^\alpha d\mu_a(u) \right) = \int_{\mathbb{R}} (1 + zu)^\alpha d\nu_a(u). \quad (28)$$

Proof. 1) Denote the left-hand side of the desired identity by $A^{-1/\theta}$ and the right-hand side by $B^{1/\alpha}$. Using the identity

$$\frac{1}{r^\theta} = \frac{1}{\Gamma(\theta)} \int_0^\infty t^{\theta-1} e^{-rt} dt,$$

we obtain

$$\begin{aligned} A &= c_{\alpha, \theta} \mathbb{E}^\alpha \left[\left(\eta(X) + z \int_X a(x) d\eta(x) \right)^{-\theta} \right] \\ &= \frac{c_{\alpha, \theta}}{\Gamma(\theta)} \mathbb{E}^\alpha \left[\int_0^\infty t^{\theta-1} \exp \left(-t \left(\eta(X) + z \int_X a(x) d\eta(x) \right) \right) dt \right] \\ &= \frac{c_{\alpha, \theta}}{\Gamma(\theta)} \int_0^\infty t^{\theta-1} \mathbb{E}^\alpha \left[\exp \left(- \int_X t(1 + za(x)) d\eta(x) \right) \right] dt. \end{aligned}$$

By the Laplace transform formula (21), the expectation equals precisely $e^{-t^\alpha B}$, thus

$$A = \frac{c_{\alpha, \theta}}{\Gamma(\theta)} \int_0^\infty t^{\theta-1} e^{-t^\alpha B} dt,$$

and (27) follows by changing variables.

2) Follows from (27) by letting $\theta \rightarrow 0$. □

Appendix

This is a continuation of the discussion on properties of the gamma process initiated in the Introduction. We will start with a new statement which is called the *subordinating property*:

6. Subordinating property [23]: There is the identity in law

$$(\beta_{\gamma_t}, t \geq 0) \stackrel{\text{law}}{=} \left(\frac{1}{\sqrt{2}} (\gamma_t^{(1)} - \gamma_t^{(2)}), t \geq 0 \right),$$

where the left-hand side, $(\beta_u, u \geq 0)$, is a Brownian motion independent from a gamma process $(\gamma_t, t \geq 0)$, and in the right-hand side $\gamma^{(1)}$ and $\gamma^{(2)}$ are two independent gamma processes.

Combining the quasi-invariance property 3 (see Introduction) and the subordinating property 6 of the gamma process, one easily deduces the following proposition, which expresses the conditional law of $(\gamma_t, t \geq 0)$ given $(X_t \equiv \beta_{\gamma_t}, t \geq 0)$, where $(\beta_u, u \geq 0)$ is a Brownian motion independent on $(\gamma_t, t \geq 0)$.

Proposition 8 ([23]) *Conditionally on X , the process $(\gamma_t, t \geq 0)$ is distributed as $\frac{1}{2}T_{V_t}^{(1)}$, $t \geq 0$, where $(T_a^{(1)}, a \geq 0)$ is the process of the first hitting times of levels $a \geq 0$ of a Brownian motion with drift 1 independent of X , and $V_t = \int_0^t |dX_s|$ is a gamma process taken at $(2t)$.*

The above Proposition suggests the existence of some deep links between the Brownian motion and the gamma process. We now present two occurrences of the gamma process related to the Brownian motion and to the Bessel processes.

Theorem 11 ([2]) *Let $(C_t, t \geq 0)$ denote the Cauchy process, and for each $x \in \mathbb{R}$ define $\mu_x = \sup\{s \leq 1 : C_s + xs = \max_{u \leq 1}(C_u + xu)\}$. Then*

$$(\mu_x, x \in \mathbb{R}) \stackrel{\text{law}}{=} (\gamma_{\rho(x)}/\gamma_1, x \in \mathbb{R}),$$

where $(\gamma_u, u \geq 0)$ is the standard gamma process, and $\rho(x) = \frac{1}{2} + \frac{1}{\pi} \arctan(x) \equiv P(C_1 \leq x)$.

See also [1, Proposition 1, p. 1535] for some occurrences of $(\gamma_u/\gamma_1, u \leq 1)$ in relation with embedded regenerative sets.

To present the second instance we have in mind, we recall that the law Q_0^δ on $C(\mathbb{R}_+, \mathbb{R}_+)$ of the square of a Bessel process of (fractional) dimension δ , i.e. of \mathbb{R}_+ -valued diffusion with infinitesimal generator $2x \frac{d^2}{dx^2} + \delta \frac{d}{dx}$, is infinitely divisible, more precisely, $Q_0^\delta Q_0^{\delta'} = Q_0^{\delta+\delta'}$ for every $\delta, \delta' \geq 0$.

In the derivation of the Lévy–Khinchin representation of Q_0^δ in [24] was shown the existence of a two-parameter process $(X_t^\delta, \delta \geq 0, t \geq 0)$ which may be described as follows:

- i) it has homogeneous independent increments in δ , with values in the space $C(\mathbb{R}_+, \mathbb{R}_+)$;
- ii) for each $\delta > 0$, $(X_t^\delta, t \geq 0)$ obeys the law Q_0^δ ;
- iii) for each $t > 0$, $(X_t^\delta, \delta \geq 0)$ is distributed as $(2t\gamma_{\delta/2}, \delta \geq 0)$.

The limit of the Bessel processes at the critical value of the parameter is similar to the situation with stable and gamma processes when α tends to zero (see Section 9).

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